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Projection operators for time-symmetrized states in Fock space

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Abstract

Projection operators associated with antiunitary time-reversal symmetry differ from the usual orthogonal projections encountered in Hilbert space. Operators that act in Fock space and project parts of a single-mode state of the electromagnetic field that can be identified as its quadratures are defined and their properties are verified by consideration of their action on single-mode coherent states. The fact that their construction involves, in an essential way, not one but two parameters is discussed and a weakened form of orthogonal projection that is specialized to a particular Fock state superposition is identified. The question of whether or not such orthogonal projections are universal is raised. In passing, it is noted that the possibility of classifying states as well as operators according to their time-reversal symmetry properties is relevant to the analysis of nonlinear wave interactions between waves with rationally related frequencies.

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1. Introduction

The possibility of representing the electric field in a classical monochromatic light wave in terms of in-phase and quadrature parts [1] implies that some such representation exists for the quantized wave. However, the treatment of the quantized wave is problematic because the identification of parts that are, respectively, even and odd functions of time cannot be achieved in the same way that even- and odd-parity parts of a wavefunction are arrived at, by orthogonal projection [2]. In the latter case the space-inversion operation, Π, defined by

$$\Pi : f(x, y, z) \mapsto f(-x, -y, -z) \tag{1}$$

which has eigenvalues +1 and -1 is connected with the projection operators P_g and P_u for the even- and odd-parity parts of a function which can be written as

$$P_g = \frac{1 + \Pi}{2} \tag{2}$$

$$P_u = \frac{1 - \Pi}{2}. \tag{3}$$

The space-inversion symmetrizing projection operators have the following properties:

$$I = P_g + P_u \qquad \text{resolution of identity}$$

$$\Pi = P_g - P_u \qquad \text{spectral resolution of } \Pi$$

$$\left.\begin{array}{l} P_g^2 = P_g \\ P_u^2 = P_u \end{array}\right\} \qquad \text{idempotency}$$

$$P_g P_u = P_u P_g = 0 \qquad \text{linear independence of subspaces}$$

and

$$\Pi P_g f(x, y, z) = P_g f(x, y, z)$$

$$\Pi P_u f(x, y, z) = -P_u f(x, y, z) \qquad \text{eigenvalue equations for } \Pi$$

$$\langle P_g f(x, y, z) | P_u f(x, y, z) \rangle = 0 \qquad \text{orthogonality.}$$

The subjects of this paper are the time-symmetrizing projection operators P_p and P_q that are related to the time-reversal operation, T, defined by

$$T : \sum_n c_n |n\rangle \mapsto \sum_n c_n^* |n\rangle \tag{4}$$

where $\sum_n c_n |n\rangle$ is a Fock space superposition. P_p and P_q project parts of a Fock state in which the expectation of the electric field is even in time (phase) or odd in time (quadrature) for some choice of time origin. These projections share some, but not all, of the properties of orthogonal projections like those for even- and odd-parity parts. The difference between space-inversion and time-reversal symmetry classifications lies in the fact that, while Π is unitary, T is antiunitary [3]. The definition of satisfactory time-symmetrizing projection operators follows the recognition that the classical phase and the quantum phase are distinct parameters which are ultimately to be fitted to the Fock state amplitudes of a particular state.

2. Projection operators for field quadratures

The time-reversal symmetry operation is not unitary. Its operator, T, is antilinear and has no spectral resolution. However, this does not prevent projection operators from being defined for the time-symmetric and time-antisymmetric parts of a quantum state that share many of the properties of the projections for even- and odd-parity described above. The symmetric part of a state has the property that in it the expectation of an observable of which the observable operator is time antisymmetric is zero. Similarly, the expectation of a time-symmetric observable in a time-antisymmetrized state is zero. In the next section projection operators that share only the following properties of orthogonal projections listed above will be defined:

$$I = P_p + P_q \qquad \text{resolution of identity}$$

$$\left.\begin{array}{l} P_p^2 = P_p \\ P_q^2 = P_q \end{array}\right\} \qquad \text{idempotency}$$

$$P_p P_q = P_q P_p = 0 \qquad \text{linear independence of subspaces.}$$

Although orthogonality and spectral resolution of the symmetry operator are absent there are additional properties that are significant in applications:

$$\langle P_p f|\hat{O}_{odd}|P_p f\rangle = 0 \qquad \text{evenness of } P_p f$$

$$\langle P_q f|\hat{O}_{even}|P_q f\rangle = 0 \qquad \text{oddness of } P_q f.$$

The mathematical basis for the existence of such operators lies in the generalization of the idea of projection. Since time-reversal, as a symmetry, does not preserve inner products, it is necessary to look for projections into subspaces of a normed space the properties of which *are* invariant under time reversal. This is possible because, although projections in linear spaces are usually introduced by way of an inner product they are actually more general in nature [4] and the norm derived from the Hilbert space inner product provides sufficient structure for them to exist. This norm, unlike the inner product itself, is invariant under time reversal.

3. Coherent states and symmetrizing projections

In order to particularize the general account given above it will be helpful to consider Fock state superpositions of a type that are quite familiar. Coherent states are Fock state superpositions of which the amplitudes can be expressed in terms of a single complex parameter α:

$$|\alpha\rangle = \exp\frac{-|\alpha|^2}{2}\sum_n \frac{\alpha^n}{(n!)^{1/2}}|n\rangle. \tag{5}$$

If the operator for the electric field at some point of an electromagnetic mode in a cavity of volume V is [5]:

$$\hat{E} = \left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)^{1/2}\vec{\varepsilon}\left\{\hat{a}\exp(-\mathrm{i}\omega t) + \hat{a}^t\exp(\mathrm{i}\omega t)\right\} \tag{6}$$

then the expectation of E at this point is

$$\begin{aligned}\langle E\rangle &= \left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)^{1/2}\vec{\varepsilon}\left\{\langle\alpha|\,\hat{a}\,|\alpha\rangle\exp(-\mathrm{i}\omega t) + \langle\alpha|\,\hat{a}^t\,|\alpha\rangle\exp(\mathrm{i}\omega t)\right\}\\ &= \left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)^{1/2}\vec{\varepsilon}\left\{\alpha\exp(-\mathrm{i}\omega t) + \alpha^*\exp(\mathrm{i}\omega t)\right\}\end{aligned} \tag{7}$$

and, if α is real, is the even function $(2\hbar\omega/\varepsilon_0 V)^{1/2}\vec{\varepsilon}\alpha\cos(\omega t)$.

Similarly, if α is imaginary the expectation value of the field is an odd function of time. More generally, the expectation value of E in a coherent state with complex parameter α is

$$\langle E\rangle = \left(\frac{2\hbar\omega}{\varepsilon_0 V}\right)^{1/2}\vec{\varepsilon}|\alpha|\cos(\omega t + \arg(\alpha)). \tag{8}$$

Consider the operators P_p and P_q which act on the Fock state superposition $\sum_n c_n|n\rangle$ thus

$$P_p : c_n \mapsto \frac{\exp\mathrm{i}\,(\beta + n\theta)}{\sin\,(\beta - n\pi/2)}\,\mathrm{Im}\left\{c_n\exp\left[-\mathrm{i}n\left(\theta + \tfrac{1}{2}\pi\right)\right]\right\} \tag{9}$$

$$P_q : c_n \mapsto \frac{-\exp\mathrm{i}n\,(\theta + \pi/2)}{\sin\,(\beta - n\pi/2)}\,\mathrm{Im}\left\{c_n\exp\left[-\mathrm{i}\,(\beta + n\theta)\right]\right\} \tag{10}$$

where Im denotes the imaginary part of what follows. When $\theta = \arg\alpha$, P_p fixes [4] a coherent state with parameter α in the sense that

$$P_p\left\{\exp(\mathrm{i}\beta)|\alpha\rangle\right\} = \exp(\mathrm{i}\beta)|\alpha\rangle \tag{11}$$

whilst P_q fixes a coherent state whose parameter is $\alpha\,\exp(\mathrm{i}\pi/2)$

$$P_q \left|\exp\left(\mathrm{i}\frac{\pi}{2}\right)\alpha\right\rangle = \left|\exp\left(\mathrm{i}\frac{\pi}{2}\right)\alpha\right\rangle. \tag{12}$$

These projection operators, which satisfy the conditions identified in the previous section, meet the requirement of projecting quantum states into parts for which the expectation values of the field are even and odd in time with respect to some choice of time origin. Thus, when the time origin is chosen in such a way that

$$\hat{E} = \left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)^{1/2} \vec{\varepsilon}\left\{\hat{a}\exp\left[-\mathrm{i}\omega(t-t_0)\right] + \hat{a}^{t}\exp\left[\mathrm{i}\omega(t-t_0)\right]\right\} \tag{13}$$

and

$$\langle E\rangle = \left(\frac{2\hbar\omega}{\varepsilon_0 V}\right)^{1/2} \vec{\varepsilon}\cos\left[\omega(t-t_0) + \arg(\alpha)\right] \tag{14}$$

then the condition $\omega t_0 = \arg(\alpha)$ ensures that the expectation value of E is an even function of time.

Freedom to choose the time origin means that a single-mode coherent state can be variously expressed as an even in time or an odd in time state according to the choice of t_0 as indicated in the argument leading up to equation (14). This freedom is removed when multimode coherent states with component fields that are rationally related in frequency are considered such as, for example, in nonlinear optics. This is because, in a multimode coherent state $|\{\alpha_i\}\rangle$, there is in general no way of choosing t_0 to satisfy the set of equations

$$\omega_i t_0 = \arg(\alpha_i) \tag{15}$$

which generalize the earlier condition for two or more modes simultaneously. In nonlinear optics where the evenness or oddness of the field superpositions has a definite influence on the phenomena that are observed [6]. For example, it may be shown by consideration of coupled mode equations that phase-matched fundamental and second harmonic waves in a nonlinear medium exchange energy only when present as a sinusoidal (odd in time) superposition.

4. Retrieval of orthogonality

The inner product, $S = \langle P_p\psi | P_q\psi\rangle$, of $P_p\psi$ and $P_q\psi$, where $\psi = \sum_n c_n|n\rangle$, has real and imaginary parts:

$$\mathrm{Re}(S) = \sum_n |c_n|^2 \cot\left(\beta - \frac{n\pi}{2}\right)\sin\left[\arg(c_n) - n\theta - \frac{n\pi}{2}\right]\sin\left[\arg(c_n) - n\theta - \beta\right] \tag{16}$$

$$\mathrm{Im}(S) = \sum_n |c_n|^2 \sin\left[\arg(c_n) - n\theta - \frac{n\pi}{2}\right]\sin\left[\arg(c_n) - n\theta - \beta\right]. \tag{17}$$

It may happen that for some choice of P_p and P_q, parametrized by θ and β, $\mathrm{Re}(S)$ and $\mathrm{Im}(S)$ vanish simultaneously. In this situation P_p and P_q are projections that are orthogonal with respect to the state ψ. The notion of orthogonality with respect to a particular data set is a familiar one in numerical analysis [7] but does not appear to have been imported into the quantum formalism before.

The question of the existence and uniqueness of simultaneous solutions of $\mathrm{Re}(S) = 0$ and $\mathrm{Im}(S) = 0$ naturally arises but has not been answered here. Instead, it has been established that for the particular set of amplitudes $\{c_n\}$ shown in table 1 the parameter values $\theta = 2.3560$ rad and $\beta = 0.3546$ rad render the projections orthogonal.

Table 1. Amplitude used to investigate simultaneous solutions of $\mathrm{Re}(S) = 0$ and $\mathrm{Im}(S) = 0$ with $\mathrm{Re}(S)$ and $\mathrm{Im}(S)$ given by equations (16) and (17).

n	$\|c_n\|$	$\arg(c_n)$ (rad)
0	0.2330	1.0409
1	0.4085	1.1190
2	0.2574	1.3691
3	0.3140	0.3834
4	0.5123	0.6046
5	0.4346	0.9281
6	0.2656	0.9284
7	0.2070	0.6815
8	0.1725	0.7194
9	0.1130	0.8358
10	0.0651	0.8268
11	0.0397	0.7546
12	0.0241	0.7635
13	0.0133	0.8019
14	0.0069	0.7982
15	0.0036	0.7759
16	0.0018	0.7784
17	0.0009	0.7907
18	0.0004	0.7894
19	0.0002	0.7824
20	0.0001	0.7832

The importance of the existence of orthogonal projections for a particular Fock space superposition lies in the fact that the metrical significance of the amplitudes in the projected components is retained as indicated by the equation

$$\langle\psi|\psi\rangle = \langle P_p\psi|P_p\psi\rangle + \langle P_q\psi|P_q\psi\rangle \tag{18}$$

just as is normally the case for quantum mechanical projections. This seems sufficient reason to regard both θ and β as significant parameters in the quadrature resolution exercise. Normally, the fact that states are represented by rays rather than vectors, means that orthogonal projection is achieved with some freedom of choice of the phase [8]. This is not the case here.

5. Conclusion

Exploration of the quantum mechanical basis for describing the electric field in a light wave as a sinusoidal or cosinusoidal superposition of fundamental and harmonics forces a generalization of the projections that are normally encountered in Hilbert space. The identification of two real parameters which together define a projection shows that, in quantum mechanics, the resolution of field quadratures entails more than just a choice of the origin of time.

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